

## LETTER TO THE EDITOR

**SU(N) Self-Dual Sine-Gordon Model and Competing Orders****P. Lecheminant<sup>1</sup> and K. Totsuka<sup>2</sup>**<sup>1</sup> Laboratoire de Physique Théorique et Modélisation, CNRS UMR 8089, Université de Cergy-Pontoise, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France<sup>2</sup> Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa Oiwake-Cho, Kyoto 606-8502, Japan

**Abstract.** We investigate the low-energy properties of a generalized quantum sine-Gordon model in one dimension with a self-dual symmetry. This model describes a class of quantum phase transitions that stems from the competition of different orders. This  $SU(N)$  self-dual sine-Gordon model is shown to be equivalent to an  $SO(N)_2$  conformal field theory perturbed by a current-current interaction, which is related to an integrable fermionic model introduced by Andrei and Destri. In the context of spin-chain problems, we give several realizations of this self-dual sine-Gordon model and discuss the universality class of the transitions.

PACS numbers: 75.10.Pq; 75.10.Jm; 71.10.Pm

*Keywords* Sine-Gordon model; Self-duality; Competing orders; Spin chains

Duality symmetries have been providing much insight in diverse areas of physics, ranging from high-energy physics to condensed matter physics or statistical mechanics. One of the main reasons for this is that a duality maps, in general, a theory in the strong coupling onto one in the weak coupling, and thus is a powerful tool for investigating strongly coupled regimes. In some lattice spin models, the duality transformation can be carried out explicitly, mapping the partition function of one theory to that of another or to the same theory if the theory is self-dual. The simplest well-known example is the Kramers-Wannier (KW) duality transformation of the two-dimensional Ising model, which even locates the critical point without calculating the partition function explicitly [1]. In the context of the equivalent one-dimensional quantum Ising model in a transverse magnetic field, this KW duality symmetry maps the weak-field (low-temperature, in 2D context) ordered phase onto the strong-field (high-temperature) paramagnetic phase and vice versa. Although the strong-field phase appears disordered, it in fact sustains a hidden order which is revealed by a disorder operator [2]. Since the disorder operator is usually non-local and dual to the standard Ising order parameter, the two phases, which are separated by the Ising critical point and characterized respectively by the order- and disorder operators, are in many respects rather different from each other. The Ising ( $Z_2$ ) quantum phase transition that occurs in this model can then be interpreted as a result of the competition between these two very different gapful orders [3].

In this letter, we shall investigate several examples of one-dimensional competing orders whose critical properties are described, in the continuum limit, by a generalization of the

quantum sine-Gordon model with a manifest self-dual symmetry. The Hamiltonian density of the model is defined by:

$$\mathcal{H}_{\text{SDSG}} = \frac{1}{2} \left[ \left( \partial_x \vec{\Phi} \right)^2 + \left( \partial_x \vec{\Theta} \right)^2 \right] - g \sum_{r \in \Delta^+} \left[ : \cos \left( \sqrt{8\pi} \vec{\alpha}_r \cdot \vec{\Phi} \right) : + : \cos \left( \sqrt{8\pi} \vec{\alpha}_r \cdot \vec{\Theta} \right) : \right], \quad (1)$$

where the summation for  $r$  is taken over the positive roots of  $\text{SU}(N)$  normalized to unity:  $\vec{\alpha}_r^2 = 1$ , and  $: \cdot :$  denotes the normal ordering symbol. The bosonic vector field  $\vec{\Phi}$  is made of  $N - 1$  free boson fields  $\Phi_a$  ( $\vec{\Phi} \equiv (\Phi_1, \dots, \Phi_{N-1})$ ) which are defined by chiral components  $\Phi_{aR,L}$  as:  $\Phi_a = \Phi_{aL} + \Phi_{aR}$ , ( $a = 1, \dots, N - 1$ ). Similarly, each component of the dual vector-field  $\vec{\Theta} = (\Theta_1, \dots, \Theta_{N-1})$  is defined by:  $\Theta_a = \Phi_{aL} - \Phi_{aR}$ . The model (1) is a generalization of the usual sine-Gordon model where we have not only cosines of  $\vec{\Phi}$  but also those of the dual field  $\vec{\Theta}$ . This field theory has been introduced in Ref. [4] for exploring critical properties of vectorial Coulomb gas models in the presence of both electric- and magnetic charges. The interacting part of the model (1) is marginal and invariant under the Gaussian duality:  $\vec{\Phi} \leftrightarrow \vec{\Theta}$  (i.e. the exchange of electric- and magnetic charges in the Coulomb gas context). In fact, as will be shown later, it has a hidden  $\text{SO}(N)$  symmetry. Nevertheless, in what follows, the model (1) will be referred to as the  $\text{SU}(N)$  self-dual sine-Gordon (SDSG) models.

This model is of a direct relevance to the problem of competing quantum orders in one dimension. Indeed, since  $\vec{\Theta}$ -field is a spatial integral of  $\partial_t \vec{\Phi}$ , two fields  $\vec{\Phi}$  and  $\vec{\Theta}$  are mutually non-local and the model (1) describes, in analogy with the above Ising duality, the competition between two completely different orders. In this respect, we shall give later several applications of the model (1). For instance, one can anticipate that it describes the competition between a generalized charge-density wave, corresponding to the vertex operator of the  $\vec{\Phi}$  field in Eq. (1), and a superconducting instability due to the perturbation depending on the dual field. The exact self-duality symmetry of the model (1) may suggest the existence of a non-trivial quantum criticality in the infrared (IR) limit that results from this competition. In the simplest case ( $N = 2$ ), the situation is well understood and a Gaussian  $\text{U}(1)$  criticality emerges whatever the sign of the coupling constant  $g$  [5, 6]. This model appears in the problem of the one-dimensional Fermi gas with backscattering and spin-non-conserving process as in the spin-1/2 XYZ Heisenberg chain [5] and also it describes critical properties of weakly-coupled Luttinger chains [7]. The low-energy property for  $N > 2$  is less clear. The perturbative study of the model (1) has been done in Refs. [4, 8] and a fixed point has been found whose nature has not been fully identified in these references.

In this letter, we shall show that for  $g < 0$ , the model (1) displays a quantum critical behavior of the level-2  $\text{SO}(N)$  Wess-Zumino-Novikov-Witten (WZNW) universality class (hereafter the level- $k$  of the Kac-Moody algebra will be denoted as  $G_k$ ) with central charge  $c = N - 1$ . In contrast, for  $g > 0$ , it has a fully gapped spectrum and is related to an integrable field theory introduced by Andrei and Destri [9].

The starting point of the solution is the introduction of  $N$  right-left moving Dirac fermions  $\Psi_{\alpha R,L}$ ,  $\alpha = 1, \dots, N$  with free-Hamiltonian density:

$$\mathcal{H}_0 = -i\Psi_{\alpha R}^\dagger \partial_x \Psi_{\alpha R} + i\Psi_{\alpha L}^\dagger \partial_x \Psi_{\alpha L}, \quad (2)$$

where the summation over repeated indexes is assumed in the following. From these Dirac fermions, one can define  $SU(N)$  “spin” chiral currents through:

$$J_{R(L)}^A = : \Psi_{\alpha R(L)}^\dagger T_{\alpha\beta}^A \Psi_{\beta R(L)} : , \quad (3)$$

where  $T^A, A = 1, \dots, N^2 - 1$  are the generators of the Lie algebra of  $SU(N)$  in the fundamental representation and normalized according to:  $\text{Tr}(T^A T^B) = \delta^{AB}/2$ . As well known, these currents satisfy the  $SU(N)_1$  Kac-Moody algebra and one can rewrite the free Hamiltonian (2) as a bilinear of currents (the so-called Sugawara form) [7, 10]:

$$\mathcal{H}_0 = \mathcal{H}_{0c} + \mathcal{H}_{0s} = \frac{\pi}{N} \left( : J_R^2 : + : J_L^2 : \right) + \frac{2\pi}{N+1} \left( : J_R^A J_R^A : + : J_L^A J_L^A : \right), \quad (4)$$

where we have introduced the  $U(1)$  “charge” currents:  $J_{R(L)} = : \Psi_{\alpha R(L)}^\dagger \Psi_{\alpha R(L)} : .$  At the level of the free theory  $\mathcal{H}_0$ , spin and charge degrees of freedom decouple and the free “spin” Hamiltonian  $\mathcal{H}_{0s}$  is nothing but that of the  $SU(N)_1$  WZNW conformal field theory (CFT). Note that the central charge of the model  $\mathcal{H}_{0s}$  is  $c = N - 1$ , i.e. the central charge of  $N - 1$  massless free bosons which describes the  $g \rightarrow 0$  limit of the SDSG model (1).

Now let us add a perturbation  $\mathcal{V}$  to the “spin” (or  $SU(N)$ ) Hamiltonian  $\mathcal{H}_{0s}$  so that the “spin” part  $\mathcal{H}_{0s} + \mathcal{V} \equiv \mathcal{H}_N$  coincide with the sine-Gordon model (1). Obviously, it should be marginal (i.e. four fermion interaction) and invariant under both chiral ( $R \leftrightarrow L$ ) symmetry and the Gaussian duality  $\vec{\Phi} \leftrightarrow \vec{\Theta}$ . This self-duality symmetry considerably restricts the form of the possible four fermion interactions. To see this, let us introduce  $N$  chiral bosonic fields  $\varphi_{\alpha R, L}$  using the Abelian bosonization of Dirac fermions [7]:

$$\Psi_{\alpha R} = \frac{\kappa_\alpha}{\sqrt{2\pi}} : \exp \left( i\sqrt{4\pi} \varphi_{\alpha R} \right) : \quad (5)$$

$$\Psi_{\alpha L} = \frac{\kappa_\alpha}{\sqrt{2\pi}} : \exp \left( -i\sqrt{4\pi} \varphi_{\alpha L} \right) : , \quad (6)$$

where the bosonic fields satisfy the commutation relation  $[\varphi_{\alpha R}, \varphi_{\beta L}] = i\delta_{\alpha\beta}/4$ . The anticommutation between fermions with different indexes is realized through the presence of Klein factors (here Majorana fermions)  $\kappa_\alpha$  with the following anticommutation rule:  $\{\kappa_\alpha, \kappa_\beta\} = 2\delta_{\alpha\beta}$ . The Gaussian duality symmetry:  $\varphi_\alpha (\equiv \varphi_{\alpha L} + \varphi_{\alpha R}) \leftrightarrow \vartheta_\alpha (\equiv \varphi_{\alpha L} - \varphi_{\alpha R})$  thus amounts to the particle-hole (P-H) transformation *only* in the right-moving (R) sector of the Dirac theory:  $\Psi_{\alpha R} \rightarrow \Psi_{\alpha R}^\dagger, \Psi_{\alpha L} \rightarrow \Psi_{\alpha L}$ . As is well known, the  $SU(N)$  generators  $T^A$  can be classified into three categories:

- Antisymmetric i.e.  $SO(N)$  part:

$$\left( T_{ij}^{SO(N)} \right)_{\alpha\beta} = -\frac{i}{2} (\delta_{i\alpha} \delta_{j\beta} - \delta_{i\beta} \delta_{j\alpha}) \quad (1 \leq i < j \leq N) \quad (7)$$

- Symmetric part:

$$\left( T_{ij}^S \right)_{\alpha\beta} = \frac{1}{2} (\delta_{i\alpha} \delta_{j\beta} + \delta_{i\beta} \delta_{j\alpha}) \quad (1 \leq i < j \leq N) \quad (8)$$

- Cartan generators (Diagonal):

$$\left( T_m^D \right)_{\alpha\beta} = \frac{1}{\sqrt{2m(m+1)}} \left( \sum_{k=1}^m \delta_{\alpha k} \delta_{\beta k} - m \delta_{\alpha, m+1} \delta_{\beta, m+1} \right), \quad (m = 1, \dots, N-1). \quad (9)$$

Since all generators belonging to the  $SO(N)$  subset are antisymmetric, we deduce that the corresponding right currents behave under the Gaussian duality as:

$$\begin{aligned} J_{R,ij}^{SO(N)} &= : \Psi_{\alpha R}^\dagger (T_{ij}^{SO(N)})_{\alpha\beta} \Psi_{\beta R} : \\ &\xrightarrow{P-H} - : \Psi_{\beta R}^\dagger (T_{ij}^{SO(N)})_{\alpha\beta} \Psi_{\alpha R} : = J_{R,ij}^{SO(N)}, \end{aligned} \quad (10)$$

whereas the remaining  $(N+2)(N-1)/2$   $SU(N)$  generators are all symmetric or diagonal and the corresponding right currents change sign under the Gaussian duality:  $J_R^{S,D} \xrightarrow{P-H} -J_R^{S,D}$ . In contrast, the Gaussian duality does nothing for  $J_L^A$ . This argument suggests that a possible model equivalent to the SDSG model (1) might be

$$\begin{aligned} \mathcal{H}_N &= \frac{2\pi}{N+1} \sum_{A \in SU(N)} \left( : J_R^A J_R^A : + : J_L^A J_L^A : \right) + \lambda \sum_{i < j}^N J_{R,ij}^{SO(N)} J_{L,ij}^{SO(N)} \\ &= \frac{4\pi}{N} \sum_{i < j}^N \left[ : (J_{R,ij}^{SO(N)})^2 : + : (J_{L,ij}^{SO(N)})^2 : \right] + \lambda \sum_{i < j}^N J_{R,ij}^{SO(N)} J_{L,ij}^{SO(N)}, \end{aligned} \quad (11)$$

for an appropriately chosen coupling constant  $\lambda$ . In fact, by using bosonization rules (5)-(6), we can derive the SDSG model (1) from (11). Plugging Eqs. (3), (5) and (6) into (11), one obtains

$$\begin{aligned} \mathcal{H}_N &= \frac{1}{N} \sum_{i < j} \left[ (\partial_x \varphi_{iR} - \partial_x \varphi_{jR})^2 + (\partial_x \varphi_{iL} - \partial_x \varphi_{jL})^2 \right] \\ &\quad - \frac{\lambda}{8\pi^2} \sum_{i < j} \left\{ : \cos(\sqrt{4\pi}(\varphi_i - \varphi_j)) : + : \cos(\sqrt{4\pi}(\vartheta_i - \vartheta_j)) : \right\}. \end{aligned} \quad (12)$$

If we introduce a charge bosonic field  $\Phi_{cR,L}$  and the  $SU(N)$  bosonic fields  $\Phi_{aR,L}$  ( $a = 1, \dots, N-1$ ) as [11]:

$$\begin{aligned} \Phi_{cR(L)} &= \frac{1}{\sqrt{N}} (\varphi_1 + \dots + \varphi_N)_{R(L)} \\ \Phi_{aR(L)} &= \frac{1}{\sqrt{a(a+1)}} (\varphi_1 + \dots + \varphi_a - a\varphi_{a+1})_{R(L)}, \end{aligned} \quad (13)$$

the non-interacting part of Eq. (12) takes the standard form of a kinetic term for free bosons and the Hamiltonian (12) reads

$$\mathcal{H}_N = \frac{1}{2} \left[ (\partial_x \vec{\Phi})^2 + (\partial_x \vec{\Theta})^2 \right] - \frac{\lambda}{8\pi^2} \sum_{r \in \Delta^+} \left\{ : \cos(\sqrt{8\pi} \vec{\alpha}_r \cdot \vec{\Phi}) : + : \cos(\sqrt{8\pi} \vec{\alpha}_r \cdot \vec{\Theta}) : \right\}. \quad (14)$$

Thus we have shown that the model (11) is indeed equivalent to the SDSG model (1) if we identify  $\lambda = 8\pi^2 g$ .

To deduce the physical properties of the model (11), it is more enlightening to introduce  $2N$  Majorana fermions  $\xi_{R,L}^i$  and  $\chi_{R,L}^i$  ( $i = 1, \dots, N$ ) from the Dirac ones:  $\Psi_{iR,L} = (\xi_{R,L}^i + i \chi_{R,L}^i) / \sqrt{2}$ . The  $J_{R(L),ij}^{SO(N)}$ , being bilinears of Dirac fermions, can be expressed in terms of the Majorana fermions:

$$J_{R(L),ij}^{SO(N)} = -\frac{i}{2} \left( \xi_{R(L)}^i \xi_{R(L)}^j + \chi_{R(L)}^i \chi_{R(L)}^j \right) = \frac{1}{2} \mathcal{J}_{R(L),ij}^{SO(N)}, \quad (15)$$

where  $\mathcal{J}_{\text{R(L)},ij}^{\text{SO}(N)}$ , being the sum of two  $\text{SO}(N)_1$  currents, is an  $\text{SO}(N)_2$  current. Therefore, we deduce that the SDSG model (1) is equivalent to the level-2  $\text{SO}(N)$  WZNW model perturbed by a marginal current-current interaction:

$$\mathcal{H}_{\text{SDSG}} = \frac{\pi}{N} \sum_{i < j} \left[ : (\mathcal{J}_{\text{R},ij}^{\text{SO}(N)})^2 : + : (\mathcal{J}_{\text{L},ij}^{\text{SO}(N)})^2 : \right] + 2\pi^2 g \sum_{i < j} \mathcal{J}_{\text{R},ij}^{\text{SO}(N)} \mathcal{J}_{\text{L},ij}^{\text{SO}(N)}. \quad (16)$$

This equation is one of the main results of this letter.

Using this equivalence, one can extract the IR properties of the SDSG model. The one-loop renormalization-group (RG) equation of the model (16) is:  $\dot{g} = (N - 2)\pi g^2$ , where  $\dot{g} = \partial g / \partial l$  ( $l$  being the RG parameter). For  $g < 0$ , the interaction is marginally irrelevant so that in the far IR limit, the model flows towards the  $\text{SO}(N)_2$  WZNW fixed point with a central charge  $c = N - 1$ . This CFT has the same central charge as the  $\text{SU}(N)_1$  WZNW model and in fact there exists a conformal embedding between them [10]:  $\text{SU}(N)_1 \supset \text{SO}(N)_2$ . In contrast, when  $g > 0$ , the interaction is marginally relevant and flows toward strong coupling. From the structure of the current-current interaction, it is naturally expected that a mass gap opens dynamically i.e. the SDSG model is a massive field theory for  $g > 0$  and  $N > 2$ . In fact, this can be explicitly shown by observing that model (16) is related to an integrable field theory introduced by Andrei and Destri [9] (see also Ref. [12]) with the following Hamiltonian:

$$\mathcal{H}_{\text{AD}} = -\frac{i}{2} \left( \bar{\psi}_{1,i} \gamma^1 \partial_x \psi_{1,i} + \bar{\psi}_{2,i} \gamma^1 \partial_x \psi_{2,i} \right) - g_{\text{AD}} (\rho^2 + \tilde{\rho}^2 + \sigma^2 + \tilde{\sigma}^2), \quad (17)$$

where  $\psi_{1,i}$  (respectively  $\psi_{2,i}$ ) is a two-component spinor formed by  $\xi_{\text{R,L}}^i$  (respectively  $\chi_{\text{R,L}}^i$ ) and  $\gamma^0 = \sigma_2$ ,  $\gamma^1 = i\sigma_1$ , and  $\gamma_5 = \sigma_3$  ( $\sigma_i$  being the Pauli matrices). The  $\text{O}(N)$ -invariant order parameters  $\rho$ ,  $\sigma$ ,  $\tilde{\rho}$  and  $\tilde{\sigma}$  are defined as:

$$\rho \equiv \frac{1}{2} \left( \bar{\psi}_{1,i} \psi_{1,i} + \bar{\psi}_{2,i} \psi_{2,i} \right) = -i \left( \xi_{\text{R}}^i \xi_{\text{L}}^i + \chi_{\text{R}}^i \chi_{\text{L}}^i \right) \quad (18)$$

$$\sigma \equiv -\bar{\psi}_{1,i} \gamma^5 \psi_{2,i} = -i \left( \xi_{\text{R}}^i \chi_{\text{L}}^i - \chi_{\text{R}}^i \xi_{\text{L}}^i \right) \quad (19)$$

$$\tilde{\rho} \equiv \frac{1}{2} \left( \bar{\psi}_{1,i} \psi_{1,i} - \bar{\psi}_{2,i} \psi_{2,i} \right) = -i \left( \xi_{\text{R}}^i \xi_{\text{L}}^i - \chi_{\text{R}}^i \chi_{\text{L}}^i \right) \quad (20)$$

$$\tilde{\sigma} \equiv \bar{\psi}_{1,i} \psi_{2,i} = -i \left( \xi_{\text{R}}^i \chi_{\text{L}}^i + \chi_{\text{R}}^i \xi_{\text{L}}^i \right). \quad (21)$$

Although  $\mathcal{H}_{\text{AD}}$  looks complicated, after bosonizing, the Hamiltonian (17) separates into two commuting pieces, a free Hamiltonian  $\mathcal{H}_{0c}$  for the massless bosonic field  $\Phi_c$  (Eq. (13)) and the  $\text{SO}(N)_2$  current-current model (16) with  $g = g_{\text{AD}}/\pi^2$ :  $\mathcal{H}_{\text{AD}} = \mathcal{H}_{0c} + \mathcal{H}_{\text{SDSG}}$ . Since the model (17) is exactly solvable by means of the Bethe ansatz [9, 12], we can extract the physical properties of  $\mathcal{H}_{\text{SDSG}}$  from the solution. The nature of the ground states may be simply understood in terms of the order parameters  $\rho$ ,  $\sigma$ ,  $\tilde{\rho}$  and  $\tilde{\sigma}$ , which form two independent  $\text{SO}(2)$  doublets  $(\rho, \sigma)$  and  $(\tilde{\rho}, \tilde{\sigma})$  [9, 13]. These two doublets are mapped onto each other by the KW duality for the Majorana fermions  $\chi^i$ :  $\chi_{\text{R}}^i \rightarrow -\chi_{\text{R}}^i$  and  $\chi_{\text{L}}^i \rightarrow \chi_{\text{L}}^i$ . From the form of the interacting part of the model (17), we readily see that it is invariant under the interchange of the two doublets. On the basis of large- $N$  semiclassical argument, the authors of Ref. [9] found that when  $g_{\text{AD}} < 0$ , this interchange symmetry is broken spontaneously in the ground state and that there are two different ground states where only one of the two doublets has a finite modulus; correspondingly massive kink excitations appear in the spectrum to connect the above ground states.

Let us now consider some physical applications of the SDSG model (1) in the context of competing orders in one dimension. As the first example, we take a spin-1 bilinear-biquadratic Heisenberg chain [14] with nearest ( $J_1$ ) and next-nearest ( $J_2$ ) interactions:

$$H_{\text{BB}} = \sum_n \sum_{a=1}^2 J_{\text{BB}}^{(a)} [\mathbf{S}_n \cdot \mathbf{S}_{n+a} + (\mathbf{S}_n \cdot \mathbf{S}_{n+a})^2] + \delta_{\text{BB}} \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1}, \quad (22)$$

with  $\mathbf{S}_n$  being a spin-1 operator at site  $n$ . The model with  $\delta_{\text{BB}} = 0$  is SU(3)-symmetric and, in particular, for  $J_{\text{BB}}^{(2)} = 0$  it reduces to an integrable model [15, 16] which displays a quantum critical behavior of the SU(3)<sub>1</sub> WZNW universality class [17, 18]. The effect of the remaining interactions can be investigated in the vicinity of the SU(3) symmetric point ( $J_{\text{BB}}^{(2)} = \delta_{\text{BB}} = 0$ ) using the low-energy approach of Ref. [18]. In fact, Itoi and Kato [18] considered a more general problem of an SU( $N$ )<sub>1</sub> WZNW model perturbed by the *most general* SO( $N$ )-symmetric marginal perturbation:

$$\mathcal{H}_{\text{IK}} = \frac{2\pi}{N+1} \left( : J_{\text{R}}^A J_{\text{R}}^A : + : J_{\text{L}}^A J_{\text{L}}^A : \right) + \lambda_1 J_{\text{R}}^A J_{\text{L}}^A + 2\lambda_2 \left( T_{\alpha\beta}^A T_{\alpha\beta}^B \right) J_{\text{R}}^B J_{\text{L}}^A, \quad (23)$$

which for  $N = 3$  should describe the low-energy physics of the SO(3) model (22) around the SU(3) symmetric point.

Using the decomposition (7,8,9) of the SU( $N$ ) generators  $T^A$ , the model (23) can be expressed in the following compact form:

$$\mathcal{H}_{\text{IK}} = \frac{2\pi}{N+1} \left( : J_{\text{R}}^A J_{\text{R}}^A : + : J_{\text{L}}^A J_{\text{L}}^A : \right) + (\lambda_1 - \lambda_2) \sum_{A \in \text{SO}(N)} J_{\text{R}}^A J_{\text{L}}^A + (\lambda_1 + \lambda_2) \sum_{A \in \text{S,D}} J_{\text{R}}^A J_{\text{L}}^A. \quad (24)$$

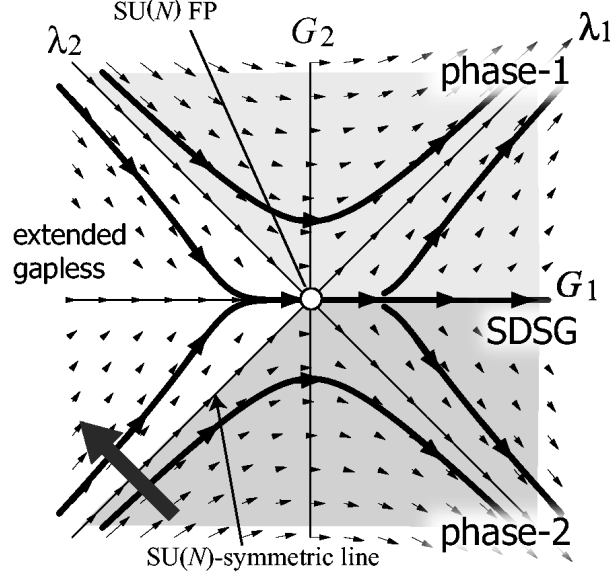
It is straightforward to calculate the one-loop RG equations [18] for the model (24) and we obtain

$$\dot{G}_1 = \frac{N-2}{8\pi} G_1^2 + \frac{N+2}{8\pi} G_2^2, \quad \dot{G}_2 = \frac{N}{4\pi} G_1 G_2, \quad (25)$$

where we have introduced a new set of couplings as  $G_1 \equiv \lambda_1 - \lambda_2$  and  $G_2 \equiv \lambda_1 + \lambda_2$ . The RG-flow is shown in Figure 1. We have two gapful phases (*phase-1* and *phase-2*) together with one extended gapless phase which belongs to the SU( $N$ )<sub>1</sub> WZNW universality class.

In the special case of  $N = 3$ , the Hamiltonian (24) describes the competition between two gapful orders of  $H_{\text{BB}}$ : a trimerization (period-3) phase (*phase-1* in Figure 1) stabilized when  $\lambda_2 = 0$  and  $\lambda_1 > 0$ , where three adjacent spins form local SU(3) singlets, and the non-degenerate Haldane state (*phase-2*) when  $\lambda_1 = 0$  and  $\lambda_2 < 0$  [18, 19]. The trimerized phase is expected to occur in  $H_{\text{BB}}$  when  $\delta_{\text{BB}} = 0$  and for a sufficiently strong value of  $J_{\text{BB}}^{(2)}$ , whereas the Haldane phase appears when  $J_{\text{BB}}^{(2)} = 0$  and  $\delta_{\text{BB}} > 0$  [14]. The one-loop RG flow is presented in Figure 1 and we see that the phase transition between these two gapful phases occurs along the line  $\lambda_1 = -\lambda_2 > 0$  shown as ‘SDSG’. From Eq. (24), we find that the effective field theory which describes the transition is given by the SO( $N$ ) current-current model (16) with  $N = 3$  and  $g = \lambda_1/4\pi^2 (= -\lambda_2/4\pi^2) > 0$ . We thus have found an example of the SDSG model (1) with  $N = 3$  which describes the competition between the Haldane and trimerized orders and corresponds to a first-order transition (since the gap opens for  $g > 0$ ).





**Figure 1.** One-loop RG flow for the Itoi-Kato model (23). In the sine-Gordon language, the  $\lambda_1$ - and  $\lambda_2$  axis respectively correspond to the pure  $\cos(\sqrt{8\pi}\vec{\alpha}_r \cdot \vec{\Phi})$  model and the pure  $\cos(\sqrt{8\pi}\vec{\alpha}_r \cdot \vec{\Theta})$  one (see Eq. (1)). Since they are related to each other by the Gaussian duality, *phase-1* and *phase-2* cannot be connected by any local symmetry. These competing gapful phases are separated by the line of self-dual sine-Gordon model (SDSG). The thick arrow schematically shows the path traced by  $H_{\text{BB}}(J_{\text{BB}}^{(2)} = 0)$  ( $H_{\text{so}}(J_{\text{so}}^{(2)} = 0)$  for  $N = 4$ ) as  $\delta_{\text{BB}}$  ( $\delta_{\text{so}}$  for  $N = 4$ ) is changed from positive (Haldane- or staggered dimerization phase) to negative (gapless  $\text{SU}(N)_1$  WZNW phase).

In the second example, we consider an  $\text{SU}(2) \times \text{SU}(2)$ -symmetric spin-orbital chain with nearest ( $J_{\text{so}}^{(1)}$ ) and next-nearest ( $J_{\text{so}}^{(2)}$ ) interactions:

$$H_{\text{so}} = \sum_n \sum_{a=1}^2 J_{\text{so}}^{(a)} \left( 2 \mathbf{S}_n \cdot \mathbf{S}_{n+a} + \frac{1}{2} \right) \left( 2 \mathbf{T}_n \cdot \mathbf{T}_{n+a} + \frac{1}{2} \right) + \delta_{\text{so}} \sum_n (\mathbf{S}_n \cdot \mathbf{S}_{n+1} + \mathbf{T}_n \cdot \mathbf{T}_{n+1}), \quad (26)$$

where  $\mathbf{S}_n$  and  $\mathbf{T}_n$  denote spin-1/2 operators representing respectively the spin- and the two-fold degenerate orbital degrees of freedom [20, 21] on the  $n$ -th site. For  $J_{\text{so}}^{(2)} = \delta_{\text{so}} = 0$ , the model coincides with an  $\text{SU}(4)$  generalization of the spin-1/2 Heisenberg chain and is exactly solvable by Bethe ansatz [16]; the model is gapless with three massless bosonic modes and the field theory describing this quantum criticality is the  $\text{SU}(4)_1$  WZNW model [17, 22] or, equivalently, the  $\text{SO}(6)_1$  WZNW theory in terms of two triplets of Majorana fermions  $\xi_{\text{R,L}}^i, \chi_{\text{R,L}}^i, i = 1, \dots, 3$  [23, 13]. Using the Majorana basis, one can derive the low-energy effective Hamiltonian of the model (26) in the vicinity of the  $\text{SU}(4)$  point ( $J_{\text{so}}^{(2)} = \delta_{\text{so}} = 0$ ). We find, using the results of Refs. [13, 23], the following effective Hamiltonian density:

$$\begin{aligned} \mathcal{H}_{\text{so}} = & -\frac{iv}{2} \left( \xi_{\text{R}}^i \partial_x \xi_{\text{R}}^i - \xi_{\text{L}}^i \partial_x \xi_{\text{L}}^i + \chi_{\text{R}}^i \partial_x \chi_{\text{R}}^i - \chi_{\text{L}}^i \partial_x \chi_{\text{L}}^i \right) \\ & + (g_1 + g_2) \left( \xi_{\text{R}}^i \xi_{\text{L}}^i + \chi_{\text{R}}^i \chi_{\text{L}}^i \right)^2 + (g_1 - g_2) \left( \xi_{\text{R}}^i \xi_{\text{L}}^i - \chi_{\text{R}}^i \chi_{\text{L}}^i \right)^2. \end{aligned} \quad (27)$$

Since the interacting part of Eq. (27) can be written as:  $\mathcal{H}_{\text{so}}^{\text{int}} = -(g_1 + g_2) \rho^2 - (g_1 - g_2) \tilde{\rho}^2$ ,

we may expect that the model describes the competition between two different fully gapped orders which are characterized, in this continuum limit, by the order parameters  $\rho$  and  $\tilde{\rho}$  of Eqs. (18,20). As has been shown in Ref. [13], they correspond respectively to an SU(4) quadrumerization (Q) phase (i.e. period 4), which is characterized by local SU(4) singlets, and to a period-2 staggered dimerization (SD) phase which is formed by alternating spin and orbital singlets [24]. In terms of the lattice coupling constants, the former phase emerges when  $\delta_{\text{so}} = 0$  and for a sufficiently large value of  $J_{\text{so}}^{(2)}$  [25] whereas the SD phase is stabilized when  $J_{\text{so}}^{(2)} = 0$  and  $\delta_{\text{so}}, J_{\text{so}}^{(1)} > 0$  [20]. The competition between these two orders can be investigated by observing that the SU(2)×SU(2)-symmetric model  $\mathcal{H}_{\text{so}}$  (26) can be recasted into the form of  $\mathcal{H}_{\text{IK}}$  (24) with  $N = 4$  since SU(2)×SU(2)~SO(4). In fact, an explicit calculation shows

$$\mathcal{H}_{\text{so}}^{\text{int}} = 8g_1 \sum_{A \in \text{SO}(4)} J_{\text{R}}^A J_{\text{L}}^A + 8g_2 \sum_{A \in \text{S,D}} J_{\text{R}}^A J_{\text{L}}^A. \quad (28)$$

With the identification  $G_{1,2} = 8g_{1,2}$  and  $N = 4$ , the RG equations (25) again describe the model  $\mathcal{H}_{\text{so}}$ . From the Figure 1, we observe that the quantum phase transition between the two competing phases (phase-1 for ‘Q’ and phase-2 for ‘SD’) occurs at  $g_2 = 0$  and  $g_1 > 0$  and is thus described by:

$$\mathcal{H}_{\text{so}}(g_2 = 0) = \mathcal{H}_{\text{IK}}(\lambda_1 = -\lambda_2 = 4g_1) = \mathcal{H}_{\text{SDSG}}^{N=4}(g = g_1/\pi^2). \quad (29)$$

From the equivalence between  $\mathcal{H}_{\text{SDSG}}$  and the massive sector of  $\mathcal{H}_{\text{AD}}$ , we conclude that the Q↔SD transition described by  $\mathcal{H}_{\text{so}}(g_2 = 0)$  is of first order.

The last example is provided by the generalized two-leg spin ladders with four-spin exchange interactions studied recently in Ref. [13]. In particular, it has been shown that, close to the SU(4) symmetric point of Eq. (26) with  $J_{\text{so}}^{(2)} = \delta_{\text{so}} = 0$ , four competing orders emerge. In addition to the Q ( $\rho$ ) and SD ( $\tilde{\rho}$ ) orders of the previous example, a scalar-chirality order [26] and a rung-quadrumerization order appear. These two additional phases are characterized, within the low-energy approach of Ref. [13], respectively by the order parameters  $\sigma$  and  $\tilde{\sigma}$  of Eqs. (19,21). As is seen from the one-loop RG analysis of Ref. [13], the competition between these four orders is governed by the low-energy effective Hamiltonian:

$$\mathcal{H}_{\text{eff}} = -\frac{iv}{2} \left( \xi_R^i \partial_x \xi_R^i - \xi_L^i \partial_x \xi_L^i + \chi_R^i \partial_x \chi_R^i - \chi_L^i \partial_x \chi_L^i \right) - \lambda \left( \rho^2 + \tilde{\rho}^2 + \sigma^2 + \tilde{\sigma}^2 \right), \quad (30)$$

with  $\lambda > 0$ . We thus observe that the Andrei-Destri model (17) with  $N = 3$  and  $g_{\text{AD}} = \lambda$  accounts for the competition between the four different orders of the problem. Since the latter model is equivalent to the SDSG model (1) up to a free massless bosonic field, we easily see that the resulting phase transition is of a U(1) Gaussian type when  $\lambda > 0$ .

We hope that other applications of the SDSG model will be reported in the near future.

## Acknowledgments

We would like to thank H. Saleur for illuminating discussions and N. Andrei, P. Azaria, E. Boulat, and A. A. Nersesyan for their interest in this work. We are also grateful to A. Läuchli for sharing his unpublished results and for useful comments on the manuscript. The author (K.T.) is grateful to the members of LPTM at Cergy-Pontoise university where



this work has been carried out. He is supported in part by the Grant-in-Aid No. 18540372 from MEXT of Japan.

## References

- [1] For a review, see Savit R, 1980 *Rev. Mod. Phys.* **52** 453.
- [2] Kadanoff L P and Ceva H, 1971 *Phys. Rev. B* **3** 3918; Fradkin E and Susskind L, 1978 *Phys. Rev. D* **17** 2637.
- [3] Sachdev S, 2000 *Science* **288** 475.
- [4] Boyanovsky B and Holman R, 1991 *Nucl. Phys. B* **358** 619.
- [5] Giamarchi T and Schulz H J, 1988 *J. Phys. (Paris)* **49** 819.
- [6] Lecheminant P, Gogolin A O and Nersesyan A A, 2002 *Nucl. Phys. B* **639** 502.
- [7] Gogolin A O, Nersesyan A A and Tsvelik A M, 1998 *Bosonization and Strongly Correlated Systems* (UK: Cambridge university press)
- [8] Sierra G and Kim E H, 2003 *J. Phys A: Math. Gen.* **36** L37.
- [9] Andrei N and Destri C, 1984 *Nucl. Phys. B* **231** 445.
- [10] Di Francesco P, Mathieu P and Sénéchal D, 1997 *Conformal Field Theory* (Berlin: Springer)
- [11] R. Assaraf, P. Azaria, M. Caffarel and P. Lecheminant, 1999 *Phys. Rev. B* **60** 2299.
- [12] Reshetikhin N, 1985 *Nucl. Phys. B* **251** 565.
- [13] Lecheminant P and Totsuka K, 2005 *Phys. Rev. B* **71** 020407; *Preprint cond-mat/0606691*.
- [14] Affleck I, Kennedy T, Lieb E H and Tasaki H, 1987 *Phys. Rev. Lett.* **59** 799.
- [15] Uimin G V, 1970 *JETP Lett.* **12** 225; Lai C K, 1974 *J. Math. Phys.* **15** 1675.
- [16] Sutherland B, 1975 *Phys. Rev. B* **12** 3795.
- [17] Affleck I, 1988 *Nucl. Phys. B* **305** 582.
- [18] Itoi C and Kato M H, 1997 *Phys. Rev. B* **55** 8295.
- [19] The ground-state degeneracy is not discussed in the original treatment [18]. We derived an effective Hamiltonian starting from the SU(3) Hubbard chain and counted the number of inequivalent (semiclassical) ground states.
- [20] Pati S K, Singh R R P and Khomskii D I, 1998 *Phys. Rev. Lett.* **81** 5406.
- [21] Yamashita Y, Shibata N and Ueda K, 1998 *Phys. Rev. B* **58** 9114.
- [22] Itoi C, Qin S and Affleck I, 2000 *Phys. Rev. B* **61** 6747.
- [23] Azaria P, Gogolin A O, Lecheminant P and Nersesyan A A, 1999 *Phys. Rev. Lett.* **83** 624; Azaria P, Boulat E and Lecheminant P, 2000 *Phys. Rev. B* **61** 12112.
- [24] Nersesyan A A and Tsvelik A M, 1997 *Phys. Rev. Lett.* **78** 3939.
- [25] Läuchli A *et al.*, *in preparation*
- [26] Läuchli A, Schmid G and Troyer M, 2003 *Phys. Rev. B* **67** 100409(R).